New integral equation form of integrable reductions of Einstein equations

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Abstract

A new development of the "monodromy transform" method for analysis of hyperbolic as well as elliptic integrable reductions of Einstein equations is presented. Compatibility conditions for some alternative representations of the fundamental solutions of associated linear systems with spectral parameter in terms of a pair of dressing ("scattering") matrices give rise to a new set of linear (quasi-Fredholm) integral equations equivalent to the symmetry reduced Einstein equations. Unlike previously derived singular integral equations constructed with the use of conserved (nonevolving) monodromy data on the spectral plane for the fundamental solutions of associated linear systems, the scalar kernels of the new equations include another kind of functional parameters – the evolving ("dynamical") monodromy data for the scattering matrices. For hyperbolic reductions, in the context of characteristic initial value problem these data are determined completely by the characteristic initial data for the fields. In terms of solutions of the new integral equations the field components are expressed in quadratures.

1 Introduction

In General Relativity in a number of physically significant cases the dynamical part of Einstein equations, being restricted to space-time geometries admitting two-dimensional Abelian isometry group[1], reduces to the nonlinear integrable systems. Among these are the Einstein equations for vacuum gravitational fields [5]–[11], the Einstein equations for space-times with a stiff matter fluid [12], the electrovacuum Einstein - Maxwell field equations [9, 10, 11, 13], the Einstein - Maxwell - Weyl equations for gravitational, electromagnetic and massless spinor fields [4], as well as some string theory induced gravity models, e.g., the Einstein - Maxwell equations with axion and dilaton fields [14] – [16]. Accordingly to the type of the contemplated two-dimensional space-time symmetry (determined by the signature of the metric on the orbit space) the reduced equations can be either of the hyperbolic or of the elliptic types. By now a theory of these equations has been developed and discussed in many aspects (for details and references see, for example, [16, 17]).

The so called monodromy transform [16, 18, 19] provides some general and fairly simple base for a description in a unified manner of all mentioned above integrable reductions of Einstein equations. Similarly to the well known inverse scattering method (the "scattering transform"), this approach begins with a representation of the dynamical part of reduced Einstein equations as the integrability conditions of an overdetermined linear system of a special structure containing a spectral parameter. The analysis of the constructed linear systems showed a common important property of the evolution of fields described by all mentioned above integrable reductions of Einstein equations. This property is a conservation of the monodromy structure on the spectral plane of the normalized fundamental solutions of associated linear systems. This monodromy structure is defined here as a set of linear transformations which relate this fundamental solution with its analytical continuations along the closed paths surrounding each of the singular points of this solution on the spectral plane. The matrices of these linear transformations possess some special algebraic structure with a small set of independent components – the functions of the spectral parameter which constitute what we call here as monodromy data.[2] A remarkable properties of the defined monodromy data are that they (i) are functions of the spectral parameter only, (ii) exist for any analytical local solution of reduced

[1] The most known and elegant form of the reduced equations is written in terms of complex potentials — the Ernst potentials. These are the vacuum Ernst equation for one complex potential [1] or electrovacuum Ernst equations, i.e. a coupled system of two similar equations for two complex Ernst potentials [2] which have been derived by F.J.Ernst originally for stationary axisymmetric fields. For the hyperbolic case these equations have very similar forms [3]. A generalized form of these equations arises, for example, in the presence of Weyl spinor field [4].

[2] For example, for any vacuum gravitational field with the supposed space-time symmetry the monodromy data consist of two functions of the spectral parameter, while for elctrovacuum fields we have four such functions.

Einstein equations and (iii) characterize uniquely every analytical local solution of the field equations. All these means, that these monodromy data can be considered as a new set of the field variables for which the field equations become trivial (being the conserved quantities, these monodromy data do not possess any evolution) or as new "coordinates" (instead of usual field variables) in the infinite dimensional spaces of local solutions of the integrable reductions of Einstein equations. Just using of such "coordinate transformation" explains why this approach was called as the monodromy transform.

In this way the "direct" and "inverse" problems of the monodromy transform suggest themselves naturally. For solution of the first of them, i.e. for calculation of the monodromy data for a given solution of reduced Einstein equations, we have to find a fundamental solution of the associated linear system with the coefficients corresponding to given field components and to determine than its monodromy data on the spectral plane. The solution of the inverse problem, i.e. the construction of a solution of Einstein equations for given monodromy data functions, reduces to the solution of a special integral equation form of the field equations derived in [18, 19]. These are the linear singular integral equations whose scalar kernels and right hand sides are expressed in terms of the mentioned above nonevolving monodromy data. (Equivalent regularizations of these equations have been derived in [20, 16, 21].) These integral equations always possess a unique solution for any given monodromy data functions holomorphic in some local regions of the spectral plane. The corresponding local solution of Einstein equations can be calculated in quadratures whose integrands are expressed in terms of the solution of the mentioned linear integral equations.

This approach suggests different applications. The first of them is a direct construction of local solutions of Einstein equations. In this case the actual problem is to chose some monodromy data which allow to solve explicitly the integral equations with the corresponding kernels and then to calculate the corresponding quadratures for the field components. Fortunately, for very large classes of the monodromy data all these can be realized, and infinite hierarchies of families of exact solutions with any finitely large number of free parameters can be calculated in elementary functions. The examples of such hierarchies are the solutions with arbitrary analytically adjusted rational monodromy data [19, 22, 23] and the solutions for colliding plane waves and inhomogeneous cosmological models with analytically not adjusted but also rational monodromy data [24]. Each of these hierarchies extends considerably the hierarchies of vacuum [6] and electrovacuum [13] multisoliton solutions provided the Minkowski space-time is chosen as the background for solitons.

Another application of the same approach is a formulation of general schemes for solution of some initial or boundary value problems for the gravitational and some matter fields with two–dimensional space-time symmetries. The basic idea is that any given initial data for a characteristic initial value problem or a Cauchy problem for integrable hyperbolic reductions of Einstein equations as well as a boundary data for some boundary problems for an elliptic (e.g.,

stationary axisymmetric) integrable reduction of Einstein equations allow to calculate the corresponding monodromy data. The conserved (i.e. nonevolving) character of these data allows to identify them with the monodromy data for the sought-for solution of the initial or boundary value problem under consideration.

In the present paper a derivation of a new form of integral equations equivalent to each of the mentioned above integrable reductions of Einstein equations is presented. For this we introduce a specific alternative representations of the fundamental solutions of the associated linear systems in terms of pairs of "scattering" matrices dressing some partial values of these fundamental solutions ("in-states"), each depending on one of two coordinates on the orbit space and on the spectral parameter. These scattering matrices were found to possess a specific analytical structure on the spectral plane, where each of them is characterized by two (algebraic in the absence of a Weyl spinor field) branchpoints and finite jump on the cut which joins these points. (It is useful to recall here that the fundamental solution of any of the associated linear systems under consideration in general possess four such branchpoints which are joined in our construction by two nonintersecting local cuts.) The consistency conditions for these alternative representations of solutions of associated linear systems give rise to linear (quasi-Fredholm) integral equations interrelating some fragments of the algebraic structures of the dressing matrices on the cuts. The scalar kernels of these equations and their right hand sides include functional parameters which characterize the monodromy properties of the dressing (scattering) matrices mentioned above. Unlike the nonevolving monodromy data for fundamental solutions of associated linear systems, these ("dynamical") monodromy data evolve and their evolution is determined by a coordinate dependence of the "in-states". In terms of solutions of these new integral equations all of the field components and the Ernst potentials, characterizing the solutions, are determined in quadratures. For hyperbolic reductions, in the context of characteristic initial value problem these "in-states" can be identified easily with the characteristic initial data for fundamental solution of associated linear system which are determined completely by the characteristic initial data for the fields.

For simplicity, we restrict all considerations following below by hyperbolic as well as elliptic reductions of vacuum Einstein equations and electrovacuum Einstein - Maxwell equations only, because there are no any principal difficulties for the realization of the constructions suggested below for all other mentioned above integrable reductions of Einstein equations.

2 Reduction of Einstein equations

In this section we describe basic definitions used below and present in unified notations the dynamical parts of reduced vacuum Einstein equations and electrovacuum Einstein - Maxwell equations in one of their most compact forms — in the form of the Ernst equations.

2.1 Metric and electromagnetic potential

The components of a 4-dimensional space-time metric and a 1-form $\underline{\Phi}$ of complex electromagnetic potential for a self-dual Maxwell 2-form in a space-time admitting 2-dimensional Abelian isometry group can be considered in the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ab} dx^a dx^b, \qquad \underline{\Phi} = \Phi_a dx^a \tag{1}$$

where $\mu, \nu, \ldots = 1, 2$; $a, b, \ldots = 3, 4$; $g_{\mu\nu}$, g_{ab} and Φ_a depend on the coordinates x^1 and x^2 only. We consider metric $g_{\mu\nu}$ locally in a conformally flat form and parametrize the nonzero field components in (1) by scalar functions

$$g_{\mu\nu} = f \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad g_{ab} = \epsilon_0 \begin{pmatrix} H & H\Omega \\ H\Omega & H\Omega^2 + \epsilon\alpha^2/H \end{pmatrix}, \quad \Phi_a = \begin{pmatrix} \Phi, & \widetilde{\Phi} \end{pmatrix} \tag{2}$$

where $f \geq 0$, $\alpha \geq 0$, $H \geq 0$; $\epsilon, \epsilon_0, \epsilon_1, \epsilon_2 = \pm 1$ are the sign symbols whose choice should provide the Lorentz signature of the metric (1). This condition is equivalent to the relation $\epsilon_1 \epsilon_2 = -\epsilon$. The sign symbol ϵ determines both, the signature of metric g_{ab} on the 2-dimensional orbits of the space-time isometry group and the signature of the conformally flat metric $g_{\mu\nu}$ on the two-dimensional orbit space of this group. We shall refer to the cases $\epsilon = 1$ and $\epsilon = -1$ as the hyperbolic and elliptic ones respectively.

The function $\alpha(x^1, x^2)$ in (2) satisfies the identity $\det \| g_{ab} \| \equiv \epsilon \alpha^2$, which means that this function characterizes a measure of area on the orbits of the isometry group. For all known integrable reductions of Einstein equations considered here $\alpha(x^1, x^2)$ is a harmonic function, and this permits to define a function β as its harmonic conjugation as follows

$$\begin{array}{l} \eta^{\mu\nu} \partial_\mu \partial_\nu \alpha = 0, \\ \partial_\mu \beta = -\epsilon \varepsilon_\mu{}^\nu \partial_\nu \alpha, \end{array} \qquad \eta^{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \qquad \varepsilon_\mu{}^\nu = \begin{pmatrix} 0 & \epsilon_1 \\ -\epsilon_2 & 0 \end{pmatrix}$$

The functions (α, β) constitute a set of "geometrically defined" coordinates which can be used instead of unspecified coordinates (x^1, x^2).[3] However, we use instead of coordinates α and β their linear combinations (ξ, η):

$$\left\{ \begin{array}{l} \xi = \beta + j\alpha \\ \eta = \beta - j\alpha \end{array} \right. \qquad \text{where} \quad j = \left\{ \begin{array}{ll} 1 & \text{for} \quad \epsilon = 1 \\ i & \text{for} \quad \epsilon = -1 \end{array} \right.$$

which are two real null cone coordinates in the hyperbolic case ($\epsilon = 1$) or complex conjugated to each other coordinates in the elliptic case ($\epsilon = -1$).

The existing gauge freedom permits without any loss of generality to impose on the metric and electromagnetic functions the normalization conditions

$$H(\xi_0, \eta_0) = 1, \quad \Omega(\xi_0, \eta_0) = 0, \quad \Phi(\xi_0, \eta_0) = 0 \tag{3}$$

where (ξ_0, η_0) are the coordinates of a chosen "reference" point P_0.

[3] In the stationary axisymmetric case these geometrically defined coordinates are known as cylindrical Weyl coordinates (ρ, z).

2.2 The Ernst equations

The dynamical part of the space-time symmetry reduced electrovacuum Einstein - Maxwell field equations for hyperbolic as well as for elliptic cases can be expressed in the form of the Ernst equations. In our notations these equations and the linear equation for α, can be written as

$$\left\{\begin{array}{l}(\mathrm{Re}\,\mathcal{E}+\overline{\Phi}\Phi)\eta^{\mu\nu}(\partial_\mu+\alpha^{-1}\partial_\mu\alpha)\partial_\nu\mathcal{E}-\eta^{\mu\nu}(\partial_\mu\mathcal{E}+2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\mathcal{E}=0\\ (\mathrm{Re}\,\mathcal{E}+\overline{\Phi}\Phi)\eta^{\mu\nu}(\partial_\mu+\alpha^{-1}\partial_\mu\alpha)\partial_\nu\Phi-\eta^{\mu\nu}(\partial_\mu\mathcal{E}+2\overline{\Phi}\partial_\mu\Phi)\partial_\nu\Phi=0\\ \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha=0\end{array}\right. \tag{4}$$

where $\mathcal{E}(x^1,x^2)$ and $\Phi(x^1,x^2)$ are complex Ernst potentials, and for vacuum $\Phi\equiv 0$. The Ernst potential $\mathcal{E}$ is defined by the expressions [1, 2]:

$$\mathrm{Re}\,\mathcal{E}=\epsilon_0 H-\overline{\Phi}\Phi,\qquad \partial_\mu(\mathrm{Im}\,\mathcal{E})=-\alpha^{-1}H^2\varepsilon_\mu{}^\nu\partial_\nu\Omega+i(\overline{\Phi}\partial_\mu\Phi-\Phi\partial_\mu\overline{\Phi}),$$

while the electromagnetic Ernst potential Φ coincides with the corresponding component of the self-dual electromagnetic potential shown in (2). The Ernst equations (4) are quasilinear equations of the hyperbolic type for $\epsilon=1$, and they are of the elliptic type for $\epsilon=-1$. In the coordinates ξ, η we put $\alpha=(\xi-\eta)/2j$ and $\eta^{\mu\nu}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ with an appropriate choice of the sign of the conformal factor f.

3 The monodromy transform

In this section we recall the basic constructions of the monodromy transform approach developed in [18, 19, 16] and applicable in a unified manner to the analysis of all integrable reductions of Einstein equations mentioned above and, in particular, to the Ernst equations (4) for vacuum and electrovacuum fields. The basic idea of this approach is the using of a specially defined functional parameters (characterizing every local solution) as new "coordinates" in the infinite dimensional space of local solutions of reduced Einstein equations instead of usual field variables. A remarkable property of these nonevolving parameters, depending on the spectral parameter only and being interpreted as the monodromy data of the fundamental solutions of associated linear systems, is that the field equations do not impose any constraints on these parameters and therefore, such "coordinate transformation" solves completely the field equations. Thus, the solution of reduced Einstein equations becomes equivalent to solution of the inverse problem of such "coordinate transform" (called as "monodromy transform"), and solution of this problem turns out to be equivalent to solution of some linear singular integral equations with scalar kernels.

3.1 Associated linear systems

Among various gauge equivalent linear systems with constant or coordinate dependent spectral parameters (see [25] for more details) we chose the Kinnersley–like linear system whose appropriately normalized fundamental solution seems to possess the most simple general analytical structure on the spectral plane. A complete set of matrix relations, equivalent to the reduced Einstein - Maxwell equations, can be expressed in terms of the four unknown matrix functions which are 2×2-matrices for vacuum gravitational fields or 3×3-matrices for gravitational and electromagnetic fields (for other integrable cases see [16]):

$$\mathbf{U}(\xi,\eta),\quad \mathbf{V}(\xi,\eta),\quad \mathbf{W}(\xi,\eta,w),\quad \mathbf{\Psi}(\xi,\eta,w) \tag{5}$$

where w is a spectral parameter and ξ, η are the defined above real (the hyperbolic case) or complex conjugated to each other (the elliptic case) coordinates.

The first group of constraints imposed on the matrix functions (5) consists of two systems of linear differential equations for $\mathbf{\Psi}$ with algebraic constraints imposed on their (also unknown) matrix coefficients

$$\left\{\begin{array}{l} 2i(w-\xi)\partial_\xi\mathbf{\Psi}=\mathbf{U}(\xi,\eta)\mathbf{\Psi}\\ 2i(w-\eta)\partial_\eta\mathbf{\Psi}=\mathbf{V}(\xi,\eta)\mathbf{\Psi}\end{array}\right. \quad\Bigg\|\quad \begin{array}{ll}\mathrm{rank}\,\mathbf{U}=1, & \mathrm{tr}\,\mathbf{U}=i\\ \mathrm{rank}\,\mathbf{V}=1, & \mathrm{tr}\,\mathbf{V}=i\end{array} \tag{6}$$

The second group of constraints provides the existence for these linear systems of a common Hermitian matrix integral of a special structure

$$\left\{\begin{array}{l}\mathbf{\Psi}^\dagger\,\mathbf{W}\,\mathbf{\Psi}=\mathbf{W}_0(w)\\ \mathbf{W}_0^\dagger(w)=\mathbf{W}_0(w)\end{array}\right. \quad\Bigg\|\quad \frac{\partial\mathbf{W}}{\partial w}=4i\mathbf{\Omega}\quad\Bigg\|\quad \mathbf{\Omega}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} \tag{7}$$

where "†" is a Hermitian conjugation such that $\mathbf{\Psi}^\dagger(\xi,\eta,w)\equiv\overline{\mathbf{\Psi}^T(\xi,\eta,\overline{w})}$ and $\mathbf{W}_0(w)$ is an arbitrary Hermitian matrix function of the spectral parameter. In a vacuum case the third rows and columns of all matrices should be omitted.[4]

The third group consists of pure gauge conditions imposed without any loss of generality on the values of $\mathbf{\Psi}$ at the chosen reference point P_0 and $\mathbf{W}_0(w)$:

$$\mathbf{\Psi}(\xi_0,\eta_0,w)=\mathbf{I},\qquad \mathbf{W}_0(w)=4i(w-\beta_0)\mathbf{\Omega}+\mathrm{diag}\,(-4\epsilon\epsilon_0\alpha_0^2,-4\epsilon_0,1) \tag{8}$$

where $\alpha_0=(\xi_0-\eta_0)/2j$ and $\beta_0=(\xi_0+\eta_0)/2$.

3.2 Field components and potentials

The conditions (5) – (8) contain all information about the specific structures of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ which these matrices should possess to be correctly expressed

[4] In all previous author's formulations of these groups of conditions, there were included also an additional condition for electrovacuum case, that the lower right element of $\mathbf{W}$ should be equal to 1. However, this condition turns out to be pure gauge one, and it can be satisfied by an appropriate choice of the normalization conditions.

in terms of the components of metric and electromagnetic potential, about the Ernst potentials and their relations to the metric and electromagnetic potential components, about the Ernst equations and all properties of the function α described above (see [19] for some explanations). So, the relations given below are not some additional constraints imposed on the introduced auxiliary functions, but they follow directly from (5) – (8). In turn, the relations, given below in this subsection, take place in general, for any local solution of the reduced electrovacuum Einstein - Maxwell equations. In particular, the matrices $\mathbf{U}$ and $\mathbf{V}$ always possess the structures

$$\begin{array}{l} \mathbf{U} = \mathcal{F}_U \cdot \widehat{\mathbf{U}} \cdot \mathcal{F}_U^{-1} \\[1ex] \mathbf{V} = \mathcal{F}_V \cdot \widehat{\mathbf{V}} \cdot \mathcal{F}_V^{-1} \end{array} \quad \Bigg\| \quad \mathcal{F}_U = \begin{pmatrix} 1 & 0 & 0 \\ p_+ & 1 & 0 \\ q_+ & 0 & 1 \end{pmatrix}, \quad \mathcal{F}_V = \begin{pmatrix} 1 & 0 & 0 \\ p_- & 1 & 0 \\ q_- & 0 & 1 \end{pmatrix}$$

where the scalar functions $p_\pm \equiv \Omega \mp \dfrac{ij\alpha}{\epsilon_0 H}$, $\quad q_\pm = 2\overline{\widetilde{\Phi}} - 2\overline{\Phi} p_\pm$ and

$$\widehat{\mathbf{U}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} i, & -\partial_\xi \mathcal{E}, & \partial_\xi \Phi \end{pmatrix} \qquad \widehat{\mathbf{V}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} i, & -\partial_\eta \mathcal{E}, & \partial_\eta \Phi \end{pmatrix}$$

Here $\mathcal{E}$ and Φ have to be identified with he Ernst potentials which characterize any local solution of reduced vacuum Einstein or electrovacuum Einstein - Maxwell equations. The matrix function $\mathbf{W}$ is linear with respect to the spectral parameter w, and its components are algebraically expressed in terms of the metric and complex electromagnetic vector potential components:

$$\mathbf{W} = 4i(w - \beta)\mathbf{\Omega} + \mathbf{G}, \qquad \mathbf{G} = \begin{pmatrix} -4h^{ab} + 4\Phi^a \overline{\Phi}^b & -2\Phi^a \\ -2\overline{\Phi}^b & 1 \end{pmatrix}$$

where $h^{ab} = \epsilon \alpha^2 g^{ab}$, g^{ab} is the matrix inverse for the metric components g_{ab}, the column $\Phi^a = (\ \widetilde{\Phi}, \quad -\Phi\ )^T$ with the superscript ^{T} meaning a transposition. Accordingly to (3) and (8), $\mathbf{W}_0(w)$ coincides with the value of $\mathbf{W}$ at the reference point (the normalization point) P_0, namely $\mathbf{W}_0(w) = \mathbf{W}(\xi_0, \eta_0, w)$.

3.3 The monodromy structure of $\mathbf{\Psi}(\xi, \eta, w)$

For any local solution of the field equations under consideration the corresponding solution of (5)–(8) for $\mathbf{\Psi}(\xi, \eta, w)$, considered as a function of w for given ξ and η running some local domains near their initial values ξ_0 and η_0 respectively, possesses a number of universal analytical properties on the spectral plane [19, 16]. In particular, the components of $\mathbf{\Psi}(\xi, \eta, w)$ and $\mathbf{\Psi}^{-1}(\xi, \eta, w)$ are holomorphic everywhere on the spectral plane (including $w = \infty$, where $\mathbf{\Psi}(\xi, \eta, w = \infty) = \mathbf{I}$) outside two nonintersecting local cuts L_+ and L_- which endpoints are two

ordered pairs of the branchpoints $(w = \xi_0, w = \xi)$ and $(w = \eta_0, w = \eta)$ respectively. In the hyperbolic case these cuts are represented by two nonoverlapping segments of the real axis on the w-plane, while in the elliptic case these cuts are located symmetrically to each other with respect to this axis in the upper and lower half-planes respectively [16]. This description of analytical properties of $\mathbf{\Psi}(\xi, \eta, w)$ we conclude presenting general expressions for the local structure of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ near the branchpoints and the cuts $L_\pm$ which join them [18, 19]:

$$
\begin{array}{ll|l}
L_+ : & \mathbf{\Psi}(\xi,\eta,w) = \lambda_+^{-1} \boldsymbol{\psi}_+(\xi,\eta,w) \otimes \mathbf{k}_+(w) + \mathbf{M}_+(\xi,\eta,w) & \multirow{2}{*}{$\lambda_+ = \sqrt{\dfrac{w-\xi}{w-\xi_0}}$} \\
 & \mathbf{\Psi}^{-1}(\xi,\eta,w) = \lambda_+ \mathbf{l}_+(w) \otimes \boldsymbol{\varphi}_+(\xi,\eta,w) + \mathbf{N}_+(\xi,\eta,w) & \\
L_- : & \mathbf{\Psi}(\xi,\eta,w) = \lambda_-^{-1} \boldsymbol{\psi}_-(\xi,\eta,w) \otimes \mathbf{k}_-(w) + \mathbf{M}_-(\xi,\eta,w) & \multirow{2}{*}{$\lambda_- = \sqrt{\dfrac{w-\eta}{w-\eta_0}}$} \\
 & \mathbf{\Psi}^{-1}(\xi,\eta,w) = \lambda_- \mathbf{l}_-(w) \otimes \boldsymbol{\varphi}_-(\xi,\eta,w) + \mathbf{N}_-(\xi,\eta,w) &
\end{array}
\tag{9}
$$

where λ_+ and λ_- are holomorphic at $w = \infty$, provided $\lambda_\pm(w = \infty) = 1$, and λ_+ and λ_- possess the jumps on L_+ and L_- respectively; all fragments of these local structures of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$, i.e. each of the row and column vectors $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$, the row and column vectors $\boldsymbol{\psi}_\pm(\xi, \eta, w)$ and $\boldsymbol{\varphi}_\pm(\xi, \eta, w)$ and the matrices $M_\pm(\xi, \eta, w)$ and $\mathbf{N}_\pm(\xi, \eta, w)$ are regular (holomorphic) on the cuts L_+ or L_- respectively their suffices, and the algebraic relations are satisfied on $L_\pm$:

$$
\begin{array}{lllll}
L_+, : & \mathbf{k}_+ \cdot \mathbf{N}_+ = 0, & \mathbf{M}_+ \cdot \mathbf{l}_+ = 0, & \boldsymbol{\varphi}_+ \cdot \mathbf{M}_+ = 0, & \mathbf{N}_+ \cdot \boldsymbol{\psi}_+ = 0 \\
L_- : & \mathbf{k}_- \cdot \mathbf{N}_- = 0, & \mathbf{M}_- \cdot \mathbf{l}_- = 0, & \boldsymbol{\varphi}_- \cdot \mathbf{M}_- = 0, & \mathbf{N}_- \cdot \boldsymbol{\psi}_- = 0
\end{array}
\tag{10}
$$

The local structure (9) with the constraints (10) allow to clarify the monodromy properties of $\mathbf{\Psi}(\xi, \eta, w)$ near its branchpoints. This structure is determined by the monodromy matrices $\mathbf{T}_+$ or $\mathbf{T}_-$ characterizing the linear transformations which relate the matrix $\mathbf{\Psi}(\xi, \eta, w)$ and its analytical continuations along, say, the clockwise directed paths t_+ and t_- joining the left edge of the cut L_+ or L_- respectively with its right edge:

$$
\mathbf{\Psi}(\xi,\eta,w) \xrightarrow{t_\pm} \mathbf{\Psi}(\xi,\eta,w) \cdot \mathbf{T}_\pm(w), \qquad \mathbf{T}_\pm(w) = \mathbf{I} - 2\frac{\mathbf{l}_\pm(w) \otimes \mathbf{k}_\pm(w)}{(\mathbf{l}_\pm(w) \cdot \mathbf{k}_\pm(w))} \tag{11}
$$

For derivation of these expressions, besides (9) and (10), we have used the properties $\lambda_+ \xrightarrow{t_+} -\lambda_+$ and $\lambda_- \xrightarrow{t_-} -\lambda_-$. One can observe easily that each of the monodromy matrices $\mathbf{T}_\pm(w)$ satisfies the identity $\mathbf{T}_\pm^2(w) \equiv \mathbf{I}$.[5]

As it was shown in [18, 19], the linear relations between $\mathbf{k}_\pm^\dagger(w)$ and $\mathbf{l}_\pm(w)$ implied by the conditions (7) allow to express the components of the monodromy

[5] We stress the point, that these properties take place for vacuum and electrovacuum cases only. These don't hold in the presence of the Weyl spinor field, when the local structures (9) become more complicate and the branchpoints can be nonalgebraic [18, 16].

matrices (11) in terms of the "projective" vectors $\mathbf{k}_\pm(w)$. These relations and the affine parametrization for $\mathbf{k}_\pm(w)$ of the form

$$\mathbf{l}_\pm(w) = 4\epsilon_0(w-\xi_0)(w-\eta_0)\mathbf{W}_0^{-1}(w)\cdot\mathbf{k}_\pm^\dagger(w), \quad \mathbf{k}_\pm(w) = \{1,\ \mathbf{u}_\pm(w),\ \mathbf{v}_\pm(w)\} \tag{12}$$

allow also to express all components of the monodromy matrices $\mathbf{T}_\pm(w)$ in terms of four scalar functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$ which we call as the monodromy data. These data are conserved, i.e. they are coordinate independent, and they can be determined (at least in principle) for any local solution of the reduced vacuum or electrovacuum field equations. For vacuum $\mathbf{v}_\pm(w) \equiv 0$ and therefore, any vacuum solution is characterized by two functions $\mathbf{u}_\pm(w)$ only.

3.4 The inverse problem of the monodromy transform

Another important property of the nonevolving monodromy data is that they characterize unambiguously every local solution. For a given monodromy data the corresponding local solution can be calculated in quadratures in terms of solution of the linear singular integral equations which scalar kernels are constructed using a given monodromy data. It is remarkable, that for any choice of the monodromy data $\mathbf{u}_+(w)$, $\mathbf{v}_+(w)$ holomorphic near the point $w=\xi_0$ and $\mathbf{u}_-(w)$, $\mathbf{v}_-(w)$ holomorphic near the point $w=\eta_0$ the solution of these singular integral equations always exists and it is unique. These integral equations, solving the inverse problem of such monodromy transform, are equivalent to the integrable reductions of Einstein equations. In [16, 21] these equations have been presented in two alternative forms and together with their equivalent regularizations. Here we recall the singular forms of these equations only:

$$\frac{1}{\pi i}\int\limits_L \frac{\mathcal{K}(\tau,\zeta)}{\zeta-\tau}\,\boldsymbol{\varphi}\,(\xi,\eta,\zeta)\,d\,\zeta = \mathbf{k}(\tau), \quad \frac{1}{\pi i}\int\limits_L \frac{\widetilde{\mathcal{K}}(\zeta,\tau)}{\zeta-\tau}\,\boldsymbol{\psi}\,(\xi,\eta,\zeta)\,d\,\zeta = \mathbf{l}(\tau) \tag{13}$$

with scalar kernels of the following structures (the arguments ξ, η are omitted):

$$\mathcal{K}(\tau,\zeta) = -[\lambda]_\zeta(\mathbf{k}(\tau)\cdot\mathbf{l}(\zeta)), \qquad \widetilde{\mathcal{K}}(\tau,\zeta) = -[\lambda^{-1}]_\zeta(\mathbf{k}(\zeta)\cdot\mathbf{l}(\tau)) \tag{14}$$

The coordinates ξ and η enter the integral equations (13) as parameters which determine the location of the endpoints of the integration paths and as arguments of the λ-multipliers in the kernels (14). The expressions $[\lambda]_\zeta$ in (14) mean the jump (i.e. a half of the difference between the values of the function on the left and right edges of a cut) of the function λ at the point $\zeta \in L$. It seems useful to recall here the way for derivation of the basic equations (13) which was suggested in [18] and to explain with more details their structure.

The described above general analytical properties of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ on the spectral plane permit to represent these matrix functions as Cauchy integrals over the cut $L = L_+ + L_-$ where the integrand densities are the jumps of these

matrix functions on L. These jumps are represented by the first terms in the right hand sides of the expressions (9), while the second terms there represent the continuous parts of these integrals determined by the principal values of the Cauchy integrals with the same densities. The integral equations (13) arise immediately, if we use the integral representations mentioned just above in the first two algebraic constraints in each line of (10).

To explain the structure of the integral equations (13), we recall that the integrals there are calculated over the cut consisting of two disconnected parts: $L = L_+ + L_-$, that each of the Cauchy type integrals in (13) splits into the sum of two integrals over L_+ and L_- respectively and only one of these two integrals is a singular one. In the integrands $[\lambda]_\zeta$ means the jump of the function $\lambda_+(\xi,\eta,\zeta)$ if $\zeta = \zeta_+ \in L_+$ or of the function $\lambda_-(\xi,\eta,\zeta)$ if $\zeta = \zeta_- \in L_-$. The unknown vector functions $\boldsymbol{\varphi}\,(\xi,\eta,\tau)$, $\boldsymbol{\psi}\,(\xi,\eta,\tau)$ and the vector functions $\mathbf{k}(\tau)$, $\mathbf{l}(\tau)$ in (13) should get the suffix "+", if their argument $\tau = \tau_+ \in L_+$ and the suffix "−", if their argument $\tau = \tau_- \in L_-$, and the corresponding suffixed vector functions should be identified with the fragments of the local structure of $\boldsymbol{\Psi}$ defined in (9). Because the parameter τ in the equations (13) also run the entire cut L, i.e. it should take the values $\tau = \tau_+ \in L_+$ as well as $\tau = \tau_- \in L_-$, each of the equations (13) represents a coupled pair of vector integral equations. More explicitly, the first of the equations (13) can be represented as a system

$$
\begin{aligned}
&\frac{1}{\pi i}\,⨍_{L_+} \frac{\mathcal{K}(\tau_+,\zeta_+)}{\zeta_+ - \tau_+}\,\boldsymbol{\varphi}_+(\zeta_+)\,d\zeta_+ + \frac{1}{\pi i}\int_{L_-} \frac{\mathcal{K}(\tau_+,\zeta_-)}{\zeta_- - \tau_+}\,\boldsymbol{\varphi}_-(\zeta_-)\,d\zeta_- = \mathbf{k}_+(\tau_+)\\
&\frac{1}{\pi i}\int_{L_+} \frac{\mathcal{K}(\tau_-,\zeta_+)}{\zeta_+ - \tau_-}\,\boldsymbol{\varphi}_+(\zeta_+)\,d\zeta_+ + \frac{1}{\pi i}\,⨍_{L_-} \frac{\mathcal{K}(\tau_-,\zeta_-)}{\zeta_- - \tau_-}\,\boldsymbol{\varphi}_-(\zeta_-)\,d\zeta_- = \mathbf{k}_-(\tau_-)
\end{aligned}
$$

with unknown vector functions $\boldsymbol{\varphi}_+(\xi,\eta,\tau_+)$, $\boldsymbol{\varphi}_-(\xi,\eta,\tau_-)$ and the scalar kernels

$$
\begin{aligned}
\mathcal{K}(\tau_\pm,\zeta_+) &= -[\lambda_+]_{\zeta_+}(\mathbf{k}_\pm(\tau_\pm)\cdot\mathbf{l}_+(\zeta_+))\\
\mathcal{K}(\tau_\pm,\zeta_-) &= -[\lambda_-]_{\zeta_-}(\mathbf{k}_\pm(\tau_\pm)\cdot\mathbf{l}_-(\zeta_-)).
\end{aligned}
$$

Similarly, the second equation in (13) leads to an equivalent system of integral equations for two unknown vector functions $\boldsymbol{\psi}_+(\xi,\eta,\tau_+)$, $\boldsymbol{\psi}_-(\xi,\eta,\tau_-)$:

$$
\begin{aligned}
&\frac{1}{\pi i}\,⨍_{L_+} \frac{\widetilde{\mathcal{K}}(\tau_+,\zeta_+)}{\zeta_+ - \tau_+}\,\boldsymbol{\psi}_+(\zeta_+)\,d\zeta_+ + \frac{1}{\pi i}\int_{L_-} \frac{\widetilde{\mathcal{K}}(\tau_+,\zeta_-)}{\zeta_- - \tau_+}\,\boldsymbol{\psi}_-(\zeta_-)\,d\zeta_- = \mathbf{l}_+(\tau_+)\\
&\frac{1}{\pi i}\int_{L_+} \frac{\widetilde{\mathcal{K}}(\tau_-,\zeta_+)}{\zeta_+ - \tau_-}\,\boldsymbol{\psi}_+(\zeta_+)\,d\zeta_+ + \frac{1}{\pi i}\,⨍_{L_-} \frac{\widetilde{\mathcal{K}}(\tau_-,\zeta_-)}{\zeta_- - \tau_-}\,\boldsymbol{\psi}_-(\zeta_-)\,d\zeta_- = \mathbf{l}_-(\tau_-)
\end{aligned}
$$

where the kernels $\widetilde{\mathcal{K}}_{\pm\pm}$ possess the forms "almost symmetric" to that of $\mathcal{K}_{\pm\pm}$:

$$
\begin{aligned}
\widetilde{\mathcal{K}}(\tau_\pm,\zeta_+) &= -[\lambda_+^{-1}]_{\zeta_+}(\mathbf{k}_+(\zeta_+)\cdot\mathbf{l}_\pm(\tau_\pm))\\
\widetilde{\mathcal{K}}(\tau_\pm,\zeta_-) &= -[\lambda_-^{-1}]_{\zeta_-}(\mathbf{k}_-(\zeta_-)\cdot\mathbf{l}_\pm(\tau_\pm)).
\end{aligned}
$$

Vector solutions $\boldsymbol{\varphi}_+(\xi,\eta,\tau_+)$, $\boldsymbol{\varphi}_-(\xi,\eta,\tau_-)$ and $\boldsymbol{\psi}_+(\xi,\eta,\tau_+)$, $\boldsymbol{\psi}_-(\xi,\eta,\tau_-)$ of these equations together with the corresponding monodromy data vectors $\mathbf{k}_+(\tau_+)$, $\mathbf{k}_-(\tau_-)$ and with expressions (12) for $\mathbf{l}_+(\tau_+)$, $\mathbf{l}_-(\tau_-)$ determine the solution of our spectral problem (5) – (8)) by means of the quadratures

$$\begin{aligned}\boldsymbol{\Psi}(\xi,\eta,w) = \mathbf{I} &+ \frac{1}{\pi i}\int\limits_{L_+} \frac{[\lambda_+^{-1}]_{\zeta_+}}{\zeta_+ - w}\,\boldsymbol{\psi}_+(\zeta_+)\otimes\mathbf{k}_+(\zeta_+)\,d\zeta_+ \\ &+ \frac{1}{\pi i}\int\limits_{L_-} \frac{[\lambda_-^{-1}]_{\zeta_-}}{\zeta_- - w}\,\boldsymbol{\psi}_-(\zeta_-)\otimes\mathbf{k}_-(\zeta_-)\,d\zeta_-\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Psi}^{-1}(\xi,\eta,w) = \mathbf{I} &+ \frac{1}{\pi i}\int\limits_{L_+} \frac{[\lambda_+]_{\zeta_+}}{\zeta_+ - w}\,\mathbf{l}_+(\zeta_+)\otimes\boldsymbol{\varphi}_+(\zeta_+)\,d\zeta_+ \\ &+ \frac{1}{\pi i}\int\limits_{L_-} \frac{[\lambda_-]_{\zeta_-}}{\zeta_- - w}\,\mathbf{l}_-(\zeta_-)\otimes\boldsymbol{\varphi}_-(\zeta_-)\,d\zeta_-\end{aligned}$$

3.5 Calculation of the field components and potentials

All components of the solution can be expressed in terms of the matrix $\mathbf{R}(\xi,\eta)$ determined by the asymptotic expansions [19]

$$\boldsymbol{\Psi} = \mathbf{I} + w^{-1}\mathbf{R} + O(w^{-2}),\quad \boldsymbol{\Psi}^{-1} = \mathbf{I} - w^{-1}\mathbf{R} + O(w^{-2}) \tag{15}$$

Hence, for this matrix we have the following alternative expressions:

$$\begin{aligned}\mathbf{R} &= \frac{1}{\pi i}\int\limits_{L_+}[\lambda_+]_{\zeta_+}\,\mathbf{l}_+(\zeta_+)\otimes\boldsymbol{\varphi}_+(\zeta_+)\,d\zeta_+ + \frac{1}{\pi i}\int\limits_{L_-}[\lambda_-]_{\zeta_-}\,\mathbf{l}_-(\zeta_-)\otimes\boldsymbol{\varphi}_-(\zeta_-)\,d\zeta_- \\ &= -\frac{1}{\pi i}\int\limits_{L_+}[\lambda_+^{-1}]_{\zeta_+}\boldsymbol{\psi}_+(\zeta_+)\otimes\mathbf{k}_+(\zeta_+)\,d\zeta_+ - \frac{1}{\pi i}\int\limits_{L_-}[\lambda_-^{-1}]_{\zeta_-}\boldsymbol{\psi}_-(\zeta_-)\otimes\mathbf{k}_-(\zeta_-)\,d\zeta_-\end{aligned}$$

The matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ and the Ernst potentials then can be expressed as follows

$$\begin{array}{ll}\mathbf{U} = 2i\partial_\xi\mathbf{R},\quad \mathbf{V} = 2i\partial_\eta\mathbf{R}, & \mathcal{E} = \epsilon_0 - 2iR_3{}^4 \\ \mathbf{W} = \mathbf{W}_0(w) - 4i(\boldsymbol{\Omega}\mathbf{R} + \mathbf{R}^\dagger\boldsymbol{\Omega}), & \Phi = 2iR_3{}^5\end{array} \tag{16}$$

where the components $R_A{}^B$ of the 3×3-matrix $\mathbf{R}$ are numbered by the index values $A, B\ldots = 3,4,5$. The corresponding expressions for the metric components g_{ab} with $a,b,\ldots = 3,4$ and nonzero components of complex electromagnetic potential Φ_a are [19]:

$$\begin{array}{l}g_{33} = \epsilon_0 - i(R_3{}^4 - \overline{R}_3{}^4) + \Phi_3\overline{\Phi}_3, \\ g_{34} = -i(\beta-\beta_0) + i(R_3{}^3 + \overline{R}_4{}^4) + \Phi_3\overline{\Phi}_4, \\ g_{44} = \epsilon_0\epsilon\alpha_0^2 + i(R_4{}^3 - \overline{R}_4{}^3) + \Phi_4\overline{\Phi}_4,\end{array}\qquad \begin{pmatrix}\Phi_3\\ \Phi_4\end{pmatrix} = 2i\begin{pmatrix}R_3{}^5\\ R_4{}^5\end{pmatrix} \tag{17}$$

4 The "integral evolution equations"

In this section we present a new way for construction of integral equations equivalent to the dynamical part of reduced Einstein equations. As in the previous section we concentrate our consideration on the vacuum Einstein equations and electrovacuum Einstein - Maxwell equations, however this construction of the integral equations can be realized for any of integrable reductions of Einstein equations mentioned in the Introduction. The structure of the derived here new equations called also as "integral evolution equations" differes essentially from the structure of singular integral equations (13) and their simplest regularizations described in various forms in [16, 21].

4.1 The "in-states" $\mathbf{\Psi}_+(\xi, w)$ and $\mathbf{\Psi}_-(\eta, w)$

As the first step we introduce two particular values $\mathbf{\Psi}_+(\xi, w)$ and $\mathbf{\Psi}_-(\eta, w)$ of the matrix function $\mathbf{\Psi}(\xi, \eta, w)$. In view of some analogy with the scattering problem we call them as "in-states". These "in-states" are defined as

$$\mathbf{\Psi}_+(\xi, w) = \mathbf{\Psi}(\xi, \eta_0, w), \qquad \mathbf{\Psi}_-(\eta, w) = \mathbf{\Psi}(\xi_0, \eta, w). \tag{18}$$

For the hyperbolic case these are the boundary values of $\mathbf{\Psi}(\xi, \eta, w)$ on the characteristics which pass through the reference point $P_0(\xi_0, \eta_0)$ in the orbit space, while for the elliptic case $\mathbf{\Psi}_+$ and $\mathbf{\Psi}_-$ are the limit values, when $\eta \to \eta_0$ or $\xi \to \xi_0$ respectively, of such analytical extension of the matrix function $\mathbf{\Psi}(\xi, \eta, w)$ which arguments ξ and η are independent complex variables instead of being complex conjugated to each other. The matrix functions (18) can be defined also as the normalized fundamental solutions of the linear ordinary systems which are the restrictions of the system (6) to the corresponding characteristics $\eta = \eta_0$ and $\xi = \xi_0$ in the orbit space (for the hyperbolic case) or the restrictions of the analytically extended system (6) to the complex surfaces $\eta = \eta_0$ and $\xi = \xi_0$ in the complexified orbit space (for the elliptic case):

$$\left\{ \begin{array}{l} 2i(w-\xi)\partial_\xi \mathbf{\Psi}_+ = \mathbf{U}(\xi, \eta_0) \cdot \mathbf{\Psi}_+ \\ \mathbf{\Psi}_+(\xi_0, w) = \mathbf{I} \end{array} \right. \qquad \left\{ \begin{array}{l} 2i(w-\eta)\partial_\eta \mathbf{\Psi}_- = \mathbf{V}(\xi_0, \eta) \cdot \mathbf{\Psi}_- \\ \mathbf{\Psi}_-(\eta_0, w) = \mathbf{I} \end{array} \right.$$

Therefore, the function $\mathbf{\Psi}_+(\xi, w)$ and its inverse are holomorphic outside L_+, while $\mathbf{\Psi}_-(\eta, w)$ and its inverse are holomorphic outside L_- and each of these matrix functions posses only two brachpoints on the spectral plane, and these branchpoints coincide with the endpoints of the cuts L_+ and L_- respectively.

On these cuts $\mathbf{\Psi}_\pm$ possess the structures similar to (9):

$$\begin{aligned} L_+ : \quad & \mathbf{\Psi}_+(\xi,w) = \lambda_+^{-1}\boldsymbol{\psi}_{0+}(\xi,w)\otimes\mathbf{k}_+(w) + \mathbf{M}_{0+}(\xi,w) \\ & \mathbf{\Psi}_+^{-1}(\xi,w) = \lambda_+\mathbf{l}_+(w)\otimes\boldsymbol{\varphi}_{0+}(\xi,w) + \mathbf{N}_{0+}(\xi,w) \\ L_- : \quad & \mathbf{\Psi}_-(\eta,w) = \lambda_-^{-1}\boldsymbol{\psi}_{0-}(\eta,w)\otimes\mathbf{k}_-(w) + \mathbf{M}_{0-}(\eta,w) \\ & \mathbf{\Psi}_-^{-1}(\eta,w) = \lambda_-\mathbf{l}_-(w)\otimes\boldsymbol{\varphi}_{0-}(\eta,w) + \mathbf{N}_{0-}(\eta,w) \end{aligned} \tag{19}$$

where λ_+ and λ_-, $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$ are the same as in (9); each of the row and column vectors $\boldsymbol{\psi}_{0\pm}$, $\boldsymbol{\varphi}_{0\pm}$ and the matrices $\mathbf{M}_{0\pm}$, $\mathbf{N}_{0\pm}$) are regular (holomorphic) on the cuts L_+ or L_- respectively their suffices, and the following algebraic constraints on the fragments of these local structures similar to (10) are satisfied on the cuts $L_\pm$:

$$\begin{array}{lllll} L_+ : & \mathbf{k}_+\cdot\mathbf{N}_{0+} = 0, & \mathbf{M}_{0+}\cdot\mathbf{l}_+ = 0, & \boldsymbol{\varphi}_{0+}\cdot\mathbf{M}_{0+} = 0, & \mathbf{N}_{0+}\cdot\boldsymbol{\psi}_{0+} = 0 \\ L_- : & \mathbf{k}_-\cdot\mathbf{N}_{0-} = 0, & \mathbf{M}_{0-}\cdot\mathbf{l}_- = 0, & \boldsymbol{\varphi}_{0-}\cdot\mathbf{M}_{0-} = 0, & \mathbf{N}_{0-}\cdot\boldsymbol{\psi}_{0-} = 0 \end{array}$$

4.2 "Scattering" matrices $\boldsymbol{\chi}_\pm(\xi,\eta,w)$ and "dynamical" monodromy data $\mathbf{m}_+(\eta,w)$ and $\mathbf{m}_-(\xi,w)$

For the next step of our construction we introduce the "dressing" or "scattering" matrices $\boldsymbol{\chi}_\pm(\xi,\eta,w)$ presenting $\mathbf{\Psi}(\xi,\eta,w)$ in two alternative forms

$$\begin{aligned} \mathbf{\Psi}(\xi,\eta,w) &= \boldsymbol{\chi}_+(\xi,\eta,w)\cdot\mathbf{\Psi}_+(\xi,w) \\ \mathbf{\Psi}(\xi,\eta,w) &= \boldsymbol{\chi}_-(\xi,\eta,w)\cdot\mathbf{\Psi}_-(\eta,w) \end{aligned} \tag{20}$$

The basic idea of such alternative representation is based on the fact that the monodromy properties of $\mathbf{\Psi}(\xi,\eta,w)$ on the spectral plane w are conserved during the evolution of the fields prescribed by the reduced Einstein equations. Therefore, the monodromy properties of $\mathbf{\Psi}(\xi,\eta,w)$ should be the same as for $\mathbf{\Psi}_+(\xi,w)$ near the cut L_+ and the same as for $\mathbf{\Psi}_-(\eta,w)$ near the cut L_-. This allows to conjecture that the matrix function $\boldsymbol{\chi}_+(\xi,\eta,w)$ should be regular on the cut L_+ and $\boldsymbol{\chi}_-(\xi,\eta,w)$ should be regular on the cut L_-. And, we have all what is necessary to check this conjecture. Namely, we know the analytical properties of $\mathbf{\Psi}$ and $\mathbf{\Psi}_\pm$ on the spectral plane and therefore, the structures of $\boldsymbol{\chi}_\pm(\xi,\eta,w)$ on the spectral plane w can be described in details. In particular, it is easy to see that each of the matrices $\boldsymbol{\chi}_\pm(\xi,\eta,w)$ is holomorphic function everywhere outside the cut $L = L_+ + L_-$. Using the expressions

$$\boldsymbol{\chi}_+(\xi,\eta,w) \equiv \mathbf{\Psi}(\xi,\eta,w)\cdot\mathbf{\Psi}_+^{-1}(\xi,\eta_0,w), \quad \boldsymbol{\chi}_-(\xi,\eta,w) \equiv \mathbf{\Psi}(\xi,\eta,w)\cdot\mathbf{\Psi}_-^{-1}(\xi_0,\eta,w)$$

we get the local structures of $\boldsymbol{\chi}_\pm$ and $\boldsymbol{\chi}_\pm^{-1}$ on L_+ in the forms

$$
\begin{aligned}
\boldsymbol{\chi}_+ &= (\mathbf{k}_+(w)\cdot\mathbf{l}_+(w))\,\boldsymbol{\psi}_+(\xi,\eta,w)\otimes\boldsymbol{\varphi}_{0+}(\xi,w)+\mathbf{M}_+(\xi,\eta,w)\cdot\mathbf{N}_{0+}(\xi,w)\\
\boldsymbol{\chi}_+^{-1} &= (\mathbf{k}_+(w)\cdot\mathbf{l}_+(w))\,\boldsymbol{\psi}_{+0}(\xi,w)\otimes\boldsymbol{\varphi}_+(\xi,\eta,w)+\mathbf{M}_{0+}(\xi,w)\cdot\mathbf{N}_+(\xi,\eta,w)\\
\boldsymbol{\chi}_- &= \big(\lambda_+^{-1}\boldsymbol{\psi}_+(\xi,\eta,w)\otimes\mathbf{k}_+(w)+\mathbf{M}_+(\xi,\eta,w)\big)\cdot\mathbf{\Psi}_-^{-1}(\eta,w)\\
\boldsymbol{\chi}_-^{-1} &= \mathbf{\Psi}_-(\eta,w)\cdot(\lambda_+\mathbf{l}_+(w)\otimes\boldsymbol{\varphi}_+(\xi,\eta,w)+\mathbf{N}_+(\xi,\eta,w))
\end{aligned}
$$

and on L_- in the similar forms:

$$
\begin{aligned}
\boldsymbol{\chi}_+ &= \big(\lambda_-^{-1}\boldsymbol{\psi}_-(\xi,\eta,w)\otimes\mathbf{k}_-(w)+\mathbf{M}_-(\xi,\eta,w)\big)\cdot\mathbf{\Psi}_+^{-1}(\xi,w)\\
\boldsymbol{\chi}_+^{-1} &= \mathbf{\Psi}_+(\xi,w)\cdot(\lambda_-\mathbf{l}_-(w)\otimes\boldsymbol{\varphi}_-(\xi,\eta,w)+\mathbf{N}_-(\xi,\eta,w))\\
\boldsymbol{\chi}_- &= (\mathbf{k}_-(w)\cdot\mathbf{l}_-(w))\,\boldsymbol{\psi}_-(\xi,\eta,w)\otimes\boldsymbol{\varphi}_{0-}(\eta,w)+\mathbf{M}_-(\xi,\eta,w)\cdot\mathbf{N}_{0-}(\eta,w)\\
\boldsymbol{\chi}_-^{-1} &= (\mathbf{k}_-(w)\cdot\mathbf{l}_-(w))\,\boldsymbol{\psi}_{0-}(\eta,w)\otimes\boldsymbol{\varphi}_-(\xi,\eta,w)+\mathbf{M}_{0-}(\eta,w)\cdot\mathbf{N}_-(\xi,\eta,w)
\end{aligned}
$$

As it was expected, the functions $\boldsymbol{\chi}_+$, $\boldsymbol{\chi}_+^{-1}$ and $\boldsymbol{\chi}_-$, $\boldsymbol{\chi}_-^{-1}$ turn out to be regular on L_+ and L_- respectively. These expressions show also that $\boldsymbol{\chi}_+$ and its inverse possess the jumps on L_- and $\boldsymbol{\chi}_-$ and its inverse possess the jumps on L_+. These jumps are degenerate matrices which rank is equal to 1 and which therefore, can be represented as the products of column and row - vectors

$$
\begin{aligned}
&[\boldsymbol{\chi}_+]_{L_-} = [\lambda_-^{-1}]\boldsymbol{\psi}_-(\xi,\eta,\tau_-)\otimes\mathbf{m}_-(\xi,\tau_-)\\
&[\boldsymbol{\chi}_-]_{L_+} = [\lambda_+^{-1}]\boldsymbol{\psi}_+(\xi,\eta,\tau_+)\otimes\mathbf{m}_+(\eta,\tau_+)\\
&[\boldsymbol{\chi}_+^{-1}]_{L_-} = [\lambda_-]\mathbf{p}_-(\xi,\tau_-)\otimes\boldsymbol{\varphi}_-(\xi,\eta,\tau_-)\\
&[\boldsymbol{\chi}_-^{-1}]_{L_+} = [\lambda_+]\mathbf{p}_+(\eta,\tau_+)\otimes\boldsymbol{\varphi}_+(\xi,\eta,\tau_+)
\end{aligned}
\tag{21}
$$

where $\tau_+\in L_+$ and $\tau_-\in L_-$. Here it is important that the vector-functions $\mathbf{m}_-(u,w)$, $\mathbf{p}_-(u,w)$ and $\mathbf{m}_+(\eta,w)$, $\mathbf{p}_+(\eta,w)$ are dependent (besides the spectral parameter $w=\tau_+$ on L_+ or $w=\tau_-$ on L_-) upon only one of two coordinates ξ and η respectively. These vector functions are determined by the expressions:

$$
\begin{aligned}
\mathbf{m}_-(\xi,w) &= \mathbf{k}_-(w)\cdot\mathbf{\Psi}_+^{-1}(\xi,w) \qquad & \mathbf{p}_-(\xi,w) &= \mathbf{\Psi}_+(\xi,w)\cdot\mathbf{l}_-(w)\\
\mathbf{m}_+(\eta,w) &= \mathbf{k}_+(w)\cdot\mathbf{\Psi}_-^{-1}(\eta,w) \qquad & \mathbf{p}_+(\eta,w) &= \mathbf{\Psi}_-(\eta,w)\cdot\mathbf{l}_+(w)
\end{aligned}
\tag{22}
$$

The components of these vector functions are determined completely by the "in-states" $\mathbf{\Psi}_+(\xi,w)$ and $\mathbf{\Psi}_-(\eta,w)$. It can be recalled here that the components of the conserved monodromy data vectors $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$ in turn are determined by $\mathbf{\Psi}_\pm$ as certain fragments of the local structures (19) of these matrix functions. It is useful to note here also that the vector functions (22) can be interpreted as the monodromy data for the scattering matrices $\boldsymbol{\chi}_\pm$ on the spectral plane. To show this we consider the monodromy matrices $\widetilde{\mathbf{T}}_\pm$ which relate, similarly

to (11), the values of $\boldsymbol{\chi}_\pm$ with their analytical continuations along the paths $t_\pm$ surrounding the branchpoints $w=\xi$ and $w=\eta$ in the clockwise directions:

$$\boldsymbol{\chi}_-(\xi,\eta,w) \xrightarrow{t_+} \boldsymbol{\chi}_-(\xi,\eta,w)\cdot\widetilde{\mathbf{T}}_+(\eta,w), \quad \boldsymbol{\chi}_+(\xi,\eta,w) \xrightarrow{t_-} \boldsymbol{\chi}_+(\xi,\eta,w)\cdot\widetilde{\mathbf{T}}_-(\xi,w)$$

The described earlier local structures of $\boldsymbol{\chi}_\pm$ on the cuts $L_\pm$ permit to derive the following expressions for these new monodromy matrices

$$\widetilde{\mathbf{T}}_+(\eta,w) = \mathbf{\Psi}_-(\eta,w)\cdot\mathbf{T}_+(w)\cdot\mathbf{\Psi}_-^{-1}(\eta,w) = \mathbf{I} - 2\frac{\mathbf{p}_+(\eta,w)\otimes\mathbf{m}_+(\eta,w)}{(\mathbf{p}_+(\eta,w)\cdot\mathbf{m}_+(\eta,w))},$$
$$\widetilde{\mathbf{T}}_-(\xi,w) = \mathbf{\Psi}_+(\xi,w)\cdot\mathbf{T}_-(w)\cdot\mathbf{\Psi}_+^{-1}(\xi,w) = \mathbf{I} - 2\frac{\mathbf{p}_-(\xi,w)\otimes\mathbf{m}_-(\xi,w)}{(\mathbf{p}_-(\xi,w)\cdot\mathbf{m}_-(\xi,w))},$$

where we have used again the properties of the functions $\lambda_\pm$, that $\lambda_+ \xrightarrow{t_+} -\lambda_+$ and $\lambda_- \xrightarrow{t_-} -\lambda_-$. Unlike (11) the matrices $\widetilde{\mathbf{T}}_\pm$ possess some evolution (i.e. a coordinate dependence), however this evolution do not violate their property to satisfy identically the relations $\widetilde{\mathbf{T}}_\pm^2 = \mathbf{I}$, which characterizes the conservation of the simple algebraic character of the branchpoints of $\boldsymbol{\chi}_\pm$. (In the presence of a Weyl spinor field the character of the branching and therefore, the expressions for the monodromy matrices $\widetilde{\mathbf{T}}_\pm$ change and the identities mentioned above do not take place. Thus, we see that the "projective" vectors $\mathbf{m}_+(\eta,w)$, $\mathbf{m}_-(\xi,w)$ and $\mathbf{p}_+(\eta,w)$, $\mathbf{p}_-(\xi,w)$ play a role of some evolving analog of the conserved monodromy data (or the scattering data) $\mathbf{k}_\pm(w)$, $\mathbf{l}_\pm(w)$ for the solutions of the system (1), and hence, we call them as "dynamical monodromy data".

4.3 Consistency of representations (20) for $\mathbf{\Psi}(\xi,\eta,w)$

Earlier it was shown, that the matrix function $\mathbf{\Psi}(\xi,\eta,w)$ is holomorphic on the w-plane outside the cut $L_+ + L_-$. Similarly, $\mathbf{\Psi}_+$ is holomorphic on the entire w-plane outside the cut L_+ and $\mathbf{\Psi}_-$ is holomorphic on the entire w-plane outside the cut L_-, and the same is true also for the inverse matrices respectively. These properties and the local structures (9) and (19) allowed us to conclude in the previous subsection that the function $\boldsymbol{\chi}_+(\xi,\eta,w)$ is regular on L_+ and possesses a jump on L_-, while the function $\boldsymbol{\chi}_-(\xi,\eta,w)$ is regular on L_- and possesses a jump on L_+. It is easy to see also that these functions should satisfy the conditions

$$\boldsymbol{\chi}_+(\xi,\eta,w=\infty) = \mathbf{I}, \qquad \boldsymbol{\chi}_-(\xi,\eta,w=\infty) = \mathbf{I}$$

This means, that we can represent $\boldsymbol{\chi}_+$ and $\boldsymbol{\chi}_-$ as the integrals over L_- and over L_+ respectively:

$$\boldsymbol{\chi}_+(\xi,\eta,w) = \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_-}\frac{[\boldsymbol{\chi}_+]_{\zeta_-}}{\zeta_- - w}\,d\zeta_- \qquad \boldsymbol{\chi}_-(\xi,\eta,w) = \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_+}\frac{[\boldsymbol{\chi}_-]_{\zeta_+}}{\zeta_+ - w}\,d\zeta_+$$

Substituting here the expressions (21) for the jumps $[\boldsymbol{\chi}_\pm]_{L_\mp}$ we get the following representations of these matrices

$$\begin{aligned}
\boldsymbol{\chi}_+(\xi,\eta,w) &= \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_-} \frac{[\lambda_-^{-1}]_{\zeta_-}}{\zeta_- - w}\boldsymbol{\psi}_-(\xi,\eta,\zeta_-)\otimes \mathbf{m}_-(\xi,\zeta_-)\,d\zeta_-\\
\boldsymbol{\chi}_-(\xi,\eta,w) &= \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_+} \frac{[\lambda_+^{-1}]_{\zeta_+}}{\zeta_+ - w}\boldsymbol{\psi}_+(\xi,\eta,\zeta_+)\otimes \mathbf{m}_+(\eta,\zeta_+)\,d\zeta_+
\end{aligned}\tag{23}$$

Similar integral representations can be derived for the inverse matrices:

$$\begin{aligned}
\boldsymbol{\chi}_+^{-1}(\xi,\eta,w) &= \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_-} \frac{[\lambda_-]_{\zeta_-}}{\zeta_- - w}\mathbf{p}_-(\xi,\zeta_-)\otimes \boldsymbol{\varphi}_-(\xi,\eta,\zeta_-)\,d\zeta_-\\
\boldsymbol{\chi}_-^{-1}(\xi,\eta,w) &= \mathbf{I} + \frac{1}{i\pi}\int\limits_{L_+} \frac{[\lambda_+]_{\zeta_+}}{\zeta_+ - w}\mathbf{p}_+(\eta,\zeta_+)\otimes \boldsymbol{\varphi}_+(\xi,\eta,\zeta_+)\,d\zeta_+
\end{aligned}\tag{24}$$

From the local representations (19) for $\tau_+ \in L_+$ and $\tau_- \in L_-$ we get:

$$\begin{aligned}
L_+ :\qquad & [\boldsymbol{\Psi}_+]_{\tau_+} = [\lambda_+^{-1}]_{\tau_+}\boldsymbol{\psi}_{0+}(\xi,\tau_+)\otimes \mathbf{k}_+(\tau_+),\\
& [\boldsymbol{\Psi}_+^{-1}]_{\tau_+} = [\lambda_+]_{\tau_+}\mathbf{l}_+(\tau_+)\otimes \boldsymbol{\varphi}_{0+}(\xi,\tau_+),\\
L_- :\qquad & [\boldsymbol{\Psi}_-]_{\tau_-} = [\lambda_-^{-1}]_{\tau_-}\boldsymbol{\psi}_{0-}(\eta,\tau_-)\otimes \mathbf{k}_-(\tau_-),\\
& [\boldsymbol{\Psi}_-^{-1}]_{\tau_-} = [\lambda_-]_{\tau_-}\mathbf{l}_-(\tau_-)\otimes \boldsymbol{\varphi}_{0-}(\eta,\tau_-).
\end{aligned}\tag{25}$$

Now we return to (20) and consider a condition of consistency of these alternative expressions for $\boldsymbol{\Psi}$. This means that we have to satisfy the condition which we present in two equivalent forms

$$\begin{aligned}
\boldsymbol{\chi}_+(\xi,\eta,w)\cdot\boldsymbol{\Psi}_+(\xi,w) &= \boldsymbol{\chi}_-(\xi,\eta,w)\cdot\boldsymbol{\Psi}_-(\eta,w)\\
\boldsymbol{\Psi}_+^{-1}(\xi,w)\cdot\boldsymbol{\chi}_+^{-1}(\xi,\eta,w) &= \boldsymbol{\Psi}_-^{-1}(\eta,w)\cdot\boldsymbol{\chi}_-^{-1}(\xi,\eta,w)
\end{aligned}\tag{26}$$

These conditions can be reduced considerably, because both sides of each of them are analytical (holomorphic) functions of the spectral parameter w everywhere on the spectral plane outside the composed cut $L = L_+ + L_-$ and both sides take the same value at $w = \infty$ which is a unit matrix. Therefore, the conditions (26) are equivalent to the condition that the jumps of the left and right hand sides of each of the equations (26) on L_+ as well as on L_- should coincide:

$$\begin{aligned}
L_+ :\qquad & \boldsymbol{\chi}_+(\tau_+)\cdot[\boldsymbol{\Psi}_+]_{\tau_+} = [\boldsymbol{\chi}_-]_{\tau_+}\cdot\boldsymbol{\Psi}_-(\tau_+)\\
& [\boldsymbol{\Psi}_+^{-1}]_{\tau_+}\cdot\boldsymbol{\chi}_+^{-1}(\tau_+) = \boldsymbol{\Psi}_-^{-1}(\tau_+)\cdot[\boldsymbol{\chi}_-^{-1}]_{\tau_+}\\
L_- :\qquad & \boldsymbol{\chi}_-(\tau_-)\cdot[\boldsymbol{\Psi}_-]_{\tau_-} = [\boldsymbol{\chi}_+]_{\tau_-}\cdot\boldsymbol{\Psi}_+(\tau_-)\\
& [\boldsymbol{\Psi}_-^{-1}]_{\tau_-}\cdot\boldsymbol{\chi}_-^{-1}(\tau_-) = \boldsymbol{\Psi}_+^{-1}(\tau_-)\cdot[\boldsymbol{\chi}_+^{-1}]_{\tau_-}
\end{aligned}\tag{27}$$

4.4 Coupled systems of "integral evolution equations"

Using of the expressions (23)– (25) in (27) leads to the following coupled pairs of the linear integral equations for different fragments of the local structures (9). One of them is a system for the vector functions $\boldsymbol{\psi}_{\pm}(\xi,\eta,w)$

$$\begin{cases} \boldsymbol{\psi}_{+}(\xi,\eta,\tau_+) - \displaystyle\int\limits_{L_-} \mathcal{S}_+(\xi,\eta,\tau_+,\zeta_-)\boldsymbol{\psi}_{-}(\xi,\eta,\zeta_-)\, d\zeta_- = \boldsymbol{\psi}_{0+}(\xi,\tau_+) \\ \boldsymbol{\psi}_{-}(\xi,\eta,\tau_-) - \displaystyle\int\limits_{L_+} \mathcal{S}_-(\xi,\eta,\tau_-,\zeta_+)\boldsymbol{\psi}_{+}(\xi,\eta,\zeta_+)\, d\zeta_+ = \boldsymbol{\psi}_{0-}(\eta,\tau_-) \end{cases} \tag{28}$$

where $\tau_+,\zeta_+ \in L_+$ and $\tau_-,\zeta_- \in L_-$, and the scalar kernels $\mathcal{S}_+$ and $\mathcal{S}_-$ are:

$$\begin{aligned} \mathcal{S}_+(\xi,\eta,\tau_+,\zeta_-) &= \frac{1}{i\pi}\frac{[\lambda_-^{-1}]_{\zeta_-}}{\zeta_- - \tau_+}\,(\mathbf{m}_-(\xi,\zeta_-)\cdot\boldsymbol{\psi}_{0+}(\xi,\tau_+)) \\ \mathcal{S}_-(\xi,\eta,\tau_-,\zeta_+) &= \frac{1}{i\pi}\frac{[\lambda_+^{-1}]_{\zeta_+}}{\zeta_+ - \tau_-}\,(\mathbf{m}_+(\eta,\zeta_+)\cdot\boldsymbol{\psi}_{0-}(\eta,\tau_-)) \end{aligned}$$

Another (alternative) system of equations which possess the vector functions $\boldsymbol{\varphi}_{\pm}(\xi,\eta,w)$ as unknown variables takes the form

$$\begin{cases} \boldsymbol{\varphi}_{+}(\xi,\eta,\tau_+) - \displaystyle\int\limits_{L_-} \widetilde{S}_+(\xi,\eta,\tau_+,\zeta_-)\boldsymbol{\varphi}_{-}(\xi,\eta,\zeta_-)\, d\zeta_- = \boldsymbol{\varphi}_{0+}(\xi,\tau_+) \\ \boldsymbol{\varphi}_{-}(\xi,\eta,\tau_-) - \displaystyle\int\limits_{L_+} \widetilde{S}_-(\xi,\eta,\tau_-,\zeta_+)\boldsymbol{\varphi}_{+}(\xi,\eta,\zeta_+)\, d\zeta_+ = \boldsymbol{\varphi}_{0-}(\eta,\tau_-) \end{cases} \tag{29}$$

where the scalar kernels $\widetilde{S}_+$ and $\widetilde{S}_-$ are:

$$\begin{aligned} \widetilde{S}_+(\xi,\eta,\tau_+,\zeta_-) &= \frac{1}{i\pi}\frac{[\lambda_-]_{\zeta_-}}{\zeta_- - \tau_+}\big(\boldsymbol{\varphi}_{0+}(\xi,\tau_+)\cdot\mathbf{p}_-(\xi,\zeta_-)\big) \\ \widetilde{S}_-(\xi,\eta,\tau_-,\zeta_+) &= \frac{1}{i\pi}\frac{[\lambda_+]_{\zeta_+}}{\zeta_+ - \tau_-}\big(\boldsymbol{\varphi}_{0-}(\eta,\tau_-)\cdot\mathbf{p}_+(\eta,\zeta_+)\big) \end{aligned}$$

The structures of the derived equations (28) or (29) may seem to be simple enough, but another form of these equations can be found useful as well.

4.5 Decoupled "integral evolution equations"

A substitution of $\boldsymbol{\psi}_{-}$ from the second of the equations (28) into the first and substitution of $\boldsymbol{\psi}_{+}$ from the first of the equations (28) into the second leads to

a pair of decoupled equations

$$\begin{aligned}
\boldsymbol{\psi}_{+}(\tau_+) - \int\limits_{L_+} \mathcal{F}_+(\tau_+,\zeta_+)\boldsymbol{\psi}_{+}(\zeta_+)\,d\zeta_+ &= \mathbf{f}_+(\tau_+)\\
\boldsymbol{\psi}_{-}(\tau_-) - \int\limits_{L_-} \mathcal{F}_-(\tau_-,\zeta_-)\boldsymbol{\psi}_{-}(\zeta_-)\,d\zeta_- &= \mathbf{f}_-(\tau_-)
\end{aligned} \tag{30}$$

where the dependence of the kernels, the right hand sides and the unknown functions on ξ and η was omitted for brevity. The kernels and right hand sides of (30) possess more complicate than in (28) and (29) structures:

$$\begin{aligned}
\mathcal{F}_+(\tau_+,\zeta_+) &= \int\limits_{L_-} \mathcal{S}_+(\tau_+,\chi_-)\mathcal{S}_-(\chi_-,\zeta_+)\,d\chi_-,\\
\mathcal{F}_-(\tau_-,\zeta_-) &= \int\limits_{L_+} \mathcal{S}_-(\tau_-,\chi_+)\mathcal{S}_+(\chi_+,\zeta_-)\,d\chi_+,\\
\mathbf{f}_+(\tau_+) &= \boldsymbol{\psi}_{0+}(\tau_+) + \int\limits_{L_-} \mathcal{S}_+(\tau_+,\chi_-)\boldsymbol{\psi}_{0-}(\chi_-)\,d\chi_-\\
\mathbf{f}_-(\tau_-) &= \boldsymbol{\psi}_{0-}(\tau_-) + \int\limits_{L_+} \mathcal{S}_-(\tau_-,\chi_+)\boldsymbol{\psi}_{0+}(\chi_+)\,d\chi_+
\end{aligned}$$

Similarly, we arrive at decoupled equations for the vector functions $\boldsymbol{\varphi}_{\pm}$:

$$\begin{aligned}
\boldsymbol{\varphi}_{+}(\tau_+) - \int\limits_{L_+} \mathcal{G}_+(\tau_+,\zeta_+)\boldsymbol{\varphi}_{+}(\zeta_+)\,d\zeta_+ &= \mathbf{g}_+(\tau_+)\\
\boldsymbol{\varphi}_{-}(\tau_-) - \int\limits_{L_-} \mathcal{G}_-(\tau_-,\zeta_-)\boldsymbol{\varphi}_{-}(\zeta_-)\,d\zeta_- &= \mathbf{g}_-(\tau_-)
\end{aligned} \tag{31}$$

which kernels and right hand sides possess the expressions

$$\begin{aligned}
\mathcal{G}_+(\tau_+,\zeta_+) &= \int\limits_{L_-} \widetilde{\mathcal{S}}_+(\tau_+,\chi_-)\widetilde{\mathcal{S}}_-(\chi_-,\zeta_+)\,d\chi_-,\\
\mathcal{G}_-(\tau_-,\zeta_-) &= \int\limits_{L_+} \widetilde{\mathcal{S}}_-(\tau_-,\chi_+)\widetilde{\mathcal{S}}_+(\chi_+,\zeta_-)\,d\chi_+,\\
\mathbf{g}_+(\tau_+) &= \boldsymbol{\varphi}_{0+}(\tau_+) + \int\limits_{L_-} \widetilde{\mathcal{S}}_+(\tau_+,\chi_-)\boldsymbol{\varphi}_{0-}(\chi_-)\,d\chi_-\\
\mathbf{g}_-(\tau_-) &= \boldsymbol{\varphi}_{0-}(\tau_-) + \int\limits_{L_+} \widetilde{\mathcal{S}}_-(\tau_-,\chi_+)\boldsymbol{\varphi}_{0+}(\chi_+)\,d\chi_+
\end{aligned}$$

For construction of any solution of reduced Einstein equations it is sufficient to solve only one of the systems of coupled vector integral equations (28) or (29)

or their decoupled vector forms (30) or (31). Besides that, all of the equations derived above, being vector equations, have scalar kernels and hence, they decouple also into separate equations for each of the vector component. In terms of their solutions all components of the solution of Einstein equations can be determined in quadratures.

4.6 Calculation of solution components

In section 3.4 we recall the expressions for the metric components g_{ab} $(a, b, \ldots = 3, 4)$, the nonzero components of a complex electromagnetic potential Φ_a as well as the matrices $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ and the Ernst potentials for any solution of reduced Einstein or Einstein - Maxwell equations in terms of the components $R_A{}^B$ $(A, B, \ldots)$ of the matrix $\mathbf{R}$ defined by the asymptotic expansion (15) of $\mathbf{\Psi}$ for $w \to \infty$ (see (16), (17) and [19]). Here we present the expressions for these components of solution in terms of the matrices $\mathbf{R}_\pm(\xi, \eta)$ defined by the asymptotic expansions of the scattering matrices

$$\boldsymbol{\chi}_\pm = \mathbf{I} + w^{-1}\mathbf{R}_\pm + O(w^{-2}), \qquad \boldsymbol{\chi}_\pm^{-1} = \mathbf{I} - w^{-1}\mathbf{R}_\pm + O(w^{-2})$$

In particular, for the matrices $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ the following expressions can be derived easily from asymptotic considerations:

$$\begin{aligned} \mathbf{U}(\xi, \eta) &= \mathbf{U}(\xi, \eta_0) + 2i\partial_\xi \mathbf{R}_+, & \mathbf{W}(\xi, \eta, w) &= \mathbf{W}(\xi, \eta_0, w) - 4i(\mathbf{\Omega}\mathbf{R}_+ + \mathbf{R}_+^\dagger \mathbf{\Omega}) \\ \mathbf{V}(\xi, \eta) &= \mathbf{V}(\xi_0, \eta) + 2i\partial_\eta \mathbf{R}_-, & &= \mathbf{W}(\xi_0, \eta, w) - 4i(\mathbf{\Omega}\mathbf{R}_- + \mathbf{R}_-^\dagger \mathbf{\Omega}) \end{aligned}$$

For calculation of other components of the solution one can use the expressions (17) where the matrix $\mathbf{R}$ defined in (15) is expressed in terms of the matrices $\mathbf{R}_\pm$. This last expression can be presented in two alternative forms:

$$\mathbf{R}(\xi, \eta) = \mathbf{R}_+(\xi, \eta) + \mathbf{R}_-(\xi, \eta_0) = \mathbf{R}_-(\xi, \eta) + \mathbf{R}_+(\xi_0, \eta).$$

To complete the present construction we present the alternative expressions for $\mathbf{R}_\pm$ which follow from the asymptotic expansions of the integral representations (23) and (24):

$$\begin{aligned} \mathbf{R}_+ &= -\frac{1}{\pi i} \int\limits_{L_-} [\lambda_-^{-1}]_{\zeta_-}\, \boldsymbol{\psi}_-(\xi, \eta, \zeta_-) \otimes \mathbf{m}_-(\xi, \zeta_-)\, d\zeta_- \\ &= \frac{1}{\pi i} \int\limits_{L_-} [\lambda_-]_{\zeta_-}\, \mathbf{p}_-(\xi, \zeta_-) \otimes \boldsymbol{\varphi}_-(\xi, \eta, \zeta_-)\, d\zeta_- \\ \mathbf{R}_- &= -\frac{1}{\pi i} \int\limits_{L_+} [\lambda_+^{-1}]_{\zeta_+}\, \boldsymbol{\psi}_+(\xi, \eta, \zeta_+) \otimes \mathbf{m}_+(\eta, \zeta_+)\, d\zeta_+ \\ &= \frac{1}{\pi i} \int\limits_{L_+} [\lambda_+]_{\zeta_+}\, \mathbf{p}_+(\eta, \zeta_+) \otimes \boldsymbol{\varphi}_+(\xi, \eta, \zeta_+)\, d\zeta_+ \end{aligned}$$

These expressions allow to calculate all components of solutions also in the alternative forms. For example, for the Ernst potentials we have

$$\begin{aligned}
\mathcal{E}(\xi,\eta) = \mathcal{E}(\xi,\eta_0) + \frac{2}{\pi}\int\limits_{L_-}[\lambda_-^{-1}]_{\zeta_-}(\mathbf{e}_1\cdot\boldsymbol{\psi}_-(\zeta_-))\;(\mathbf{m}_-(\zeta_-)\cdot\mathbf{e}_2)\,d\zeta_- = \\
= \mathcal{E}(\xi_0,\eta) + \frac{2}{\pi}\int\limits_{L_+}[\lambda_+^{-1}]_{\zeta_+}(\mathbf{e}_1\cdot\boldsymbol{\psi}_+(\zeta_+))\;(\mathbf{m}_+(\zeta_+)\cdot\mathbf{e}_2)\,d\zeta_- \\
\Phi(\xi,\eta) = \Phi(\xi,\eta_0) - \frac{2}{\pi}\int\limits_{L_-}[\lambda_-^{-1}]_{\zeta_-}(\mathbf{e}_1\cdot\boldsymbol{\psi}_-(\zeta_-))\;(\mathbf{m}_-(\zeta_-)\cdot\mathbf{e}_3)\,d\zeta_- = \\
= \Phi(\xi_0,\eta) - \frac{2}{\pi}\int\limits_{L_+}[\lambda_+^{-1}]_{\zeta_+}(\mathbf{e}_1\cdot\boldsymbol{\psi}_+(\zeta_+))\;(\mathbf{m}_+(\zeta_+)\cdot\mathbf{e}_3)\,d\zeta_-
\end{aligned}$$

where $\mathbf{e}_1 = \{1,0,0\}$, $\mathbf{e}_2 = \{0,1,0\}$ and $\mathbf{e}_3 = \{0,0,1\}$.

5 Concluding remarks

In this paper the two-dimensional space-time symmetry reductions of vacuum Einstein equations and electrovacuum Einstein - Maxwell equations have been presented in some new linear (quasi-Fredholm) integral equation forms. Similar equations can be derived for all other known integrable reductions of Einstein equations using the same method without any its essential modifications.

The alternative representations (20) of the solution $\mathbf{\Psi}$ of associated linear system in terms of "in-states" $\mathbf{\Psi}_\pm$ and the scattering matrices $\boldsymbol{\chi}_\pm$ used here may be considered as some analogue of the well known dressing methods, developed for solution of various completely integrable systems (see [26, 27, 6] and the references there). In some points the present construction can remind also Krichever's construction [28] of the "analogue of d'Alembert formula" for the Sine-Gordon equation, as well as a construction of the homogeneous Hilbert problem and the matrix linear integral equation form of vacuum Ernst equation presented by Hauser and Ernst [3]. In particular, in [3] the matrices identical to $\mathbf{\Psi}_\pm$ had been used as important elements of the developed method. However, unlike the mentioned above constructions, closely related with formulations of various matrix Riemann or Riemann - Hilbert problems, the present analysis is based on a more detail consideration of some features of the structure of reduced Einstein equations which allow to reduce the problem to much more simple, scalar quasi-Fredholm integral equations.

In comparison with the previously derived (in the framework of the same, monodromy transform approach) singular integral equation form of reduced Einstein equations or their regularizations, which were expressed in terms of conserved monodromy data [18, 19], the new integral equations are designed mainly for consideration of initial and boundary value problems for the reduced

Einstein equatons. Though the “old” equations had already provided us with a principle scheme for solution of the characteristic initial value problems and the Cauchy problems in the hyperbolic cases and some boundary problems in the elliptic cases of integrable reductions of Einstein equations [19, 29], the new integral equations are obviously better adapted at least for solution of characteristic initial value problems. The scalar kernels and the right hand sides of the new equations carry more explicit information about characteristic initial data, because they are expressed in terms of “dynamical” monodromy data evolving along the characteristics and being determined completely in terms of a given characteristic initial data for the fields.

It is interesting to note, that the applicability of the presented here the “integral evolution equation” form of reduced Einstein equations is restricted from the beginning by the condition that the analytical (in terms of the coordinates ξ and η) local (i.e. near the point where the boundary characteristics intersect) solutions are considered only. This restriction leaves out of our consideration some kinds of characteristic initial value problems which correspond to physically interesting enough situations such as, for example, a collision of plane gravitational waves propagating with distinct wavefronts on the Minkowski background. In this case, the regularity of solutions which take place in some globally defined coordinates is not compatible with their analyticity in terms of the “geometrically defined coordinates” ξ and η near the point of the wavefronts collision [30]. Therefore, to consider a collision of such waves we need to refine essentially on our methods. Fortunately the monodromy transform approach, which farther development was presented in this paper, admits an appropriate generalization. This generalization was found very recently in the author’s collaborated paper with J.B.Griffiths [31] where the generalized system of linear “integral evolution equations” was derived and a method for direct solving of the characteristic initial value problem for given characteristic initial data for colliding plane gravitational or gravitational and electromagnetic waves propagating with distinct wavefronts on the Minkowski background was presented.

It is necessary to note here also, that in the cases of a Cauchy problem for hyperbolic reductions or boundary problems for elliptic reductions of Einstein equations, the interrelations between the initial or boundary data for the fields and the functional parameters in the kernels and coefficients of the integral equations derived in the present paper turn out to be more complicate than in the characteristic case. Therefore, for solutions of these problems there is no yet a similar more simple way than suggested in [19, 29] general scheme for consideration of such problems using the linear singular integral equations whose construction is based on the conserved monodormy data.

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